Week 8.

Summary:

Goal: Study vector subspace in $\mathbb{R}^n$.

(I): $V \subseteq \mathbb{R}^n$ is vector subspace if

① $V \neq \phi$

② $\forall x, y \in V$, $x+y \in V$ (closed under addition)

③ $\forall x \in V$, $\alpha \in \mathbb{R}$, $\alpha x \in V$ (closed under scaler multiplication)

Eg: ① $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; ax+by+cz=0 \right\}$ where a, b, c are some given Real numbers

checking: ① $V \neq \phi$ since $0 \in V$

② if $\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \in V$,

then $\begin{bmatrix} x+x' \\ y+y' \\ z+z' \end{bmatrix} \in V$ since

$$a(x+x') + b(y+y') + c(z+z') = (ax+by+cz) + (ax'+by'+cz') = 0.$$

③ if $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V$, $\alpha \in \mathbb{R}$, then

$$a(\alpha x) + b(\alpha y) + c(\alpha z) = \alpha(ax+by+cz) = \alpha \cdot 0 = 0 \quad \#$$

(B) (Not important construction)

Given $S = \{u_1, \cdots, u_m\} \subseteq \mathbb{R}^n$,

$V = \text{span}(S) = \{ \sum_{i=1}^{n} \alpha_i u_i \mid \alpha_i \in \mathbb{R} \}$ is a vector subspace of $\mathbb{R}^n$

Checking: ① $0 \in V \Rightarrow V \neq \phi$

② if $x \in V$, $y \in V$

$$\begin{cases} x = \sum_{i=1}^{n} \alpha_i u_i & \text{for some } \alpha_i \in \mathbb{R} \\ y = \sum_{i=1}^{n} \tilde{\alpha}_i u_i & \text{for some other } \tilde{\alpha}_i \in \mathbb{R} \end{cases}$$

then $x+y = \sum_{i=1}^{n} (\alpha_i + \tilde{\alpha}_i) u_i = \sum_{i=1}^{n} \beta_i u_i$

where $\beta_i = \alpha_i + \tilde{\alpha}_i \in \mathbb{R}$

$\therefore x+y \in V$

③ if $x \in V$, $\alpha \in \mathbb{R}$, $\Rightarrow x = \sum_{i=1}^{n} \alpha_i u_i$ for some $\alpha_i \in \mathbb{R}$

then $\alpha x = \sum_{i=1}^{n} (\alpha \alpha_i) u_i = \sum_{i=1}^{n} \beta_i u_i \in V$.

Rmk: some quick checking for non-subspace

① $0 \in V$? ② $x \in V \overset{?}{\Rightarrow} -x \in V$

Non-example (for better idea)

ⓐ $V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; x+y+z+w = 1 \right\}$ $\because 0 \notin V$

ⓑ $V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; x = yzw \right\}$

$\because \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in V$ But $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \notin V$. (although $0 \in V$)

ⓒ $V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \;\middle|\; x^2 + y^2 = z^2 + w^2 \right\}$ ① $0 \in V$ ② scalar multiplication is closed

But not closed under addition

· $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in V$, $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \notin V$. just some random check.

ⓓ $V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : xyzw = 0 \right\}$

· $0 \in V$ · scalar multiplication is closed

$\boxed{\text{But}}$ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in V$

and $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \notin V$.

Relation to matrix multiplication :

given a $m \times n$ matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix}$

where $a_{ij} \in \mathbb{R}$, we write A as

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{array}\right] = [u_1 | u_2 \cdots | u_n]$$

where u_i is a $m \times 1$ column vector (or element in $\mathbb{R}^m$);

given $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$,

$$Ax = [u_1 | u_2 | \cdots | u_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} \overset{\text{coeff}}{\underline{x_i}}\, \overset{\text{vector}}{u_i} \in \mathbb{R}^m \in \operatorname{span}(u_1, \cdots, u_n)$$

write $\operatorname{Span}\{u_1, \cdots, u_n\}$ as $C(A)$ = column space of A.

$$= \{ y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n, Ax = y \}$$

sometimes more convenient

Thm: Let H be $m \times n$ matrix,
let G be $m \times k$ matrix,
let A be $m \times m$ non-singular matrix.

Then $C(AG) = C(AH)$ iff $C(G) = C(H)$.

pf: $(\Rightarrow)$ if $C(AG) = C(AH)$

let $y \in C(G)$, $\exists x \in \mathbb{R}^k$ s.t. $Gx = y \in \mathbb{R}^m$

$\Rightarrow AGx = Ay \in C(AG) = C(AH)$.

$\therefore\ \exists\ \tilde{x} \in \mathbb{R}^k$ s.t. $Ay = AH\tilde{x}$

$$\Rightarrow y = A^{-1}Ay = A^{-1}(AH\tilde{x}) = H\tilde{x}$$

$$\Rightarrow y \in C(H)$$

$\therefore\ C(G) \subseteq C(H).$

Interchanging G and H $\Rightarrow C(H) \subseteq C(G)$ #

$(\Leftarrow)$: if $C(H) = C(G)$.

let $y \in C(AH)$, $\exists\ x \in \mathbb{R}^n$ s.t. $AHx = y$

$$\because\ Hx \in C(H) = C(G)$$

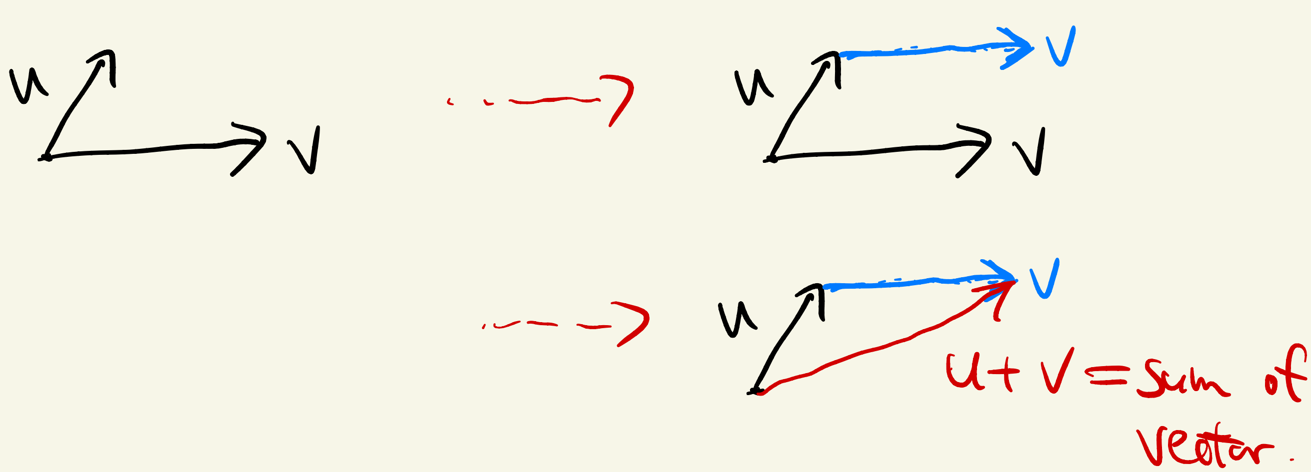
$$\therefore\ \exists\ \tilde{x} \in \mathbb{R}^k \text{ s.t. } Hx = G\tilde{x}$$

$$\therefore\ y = AG\tilde{x} \in C(AG). \Rightarrow C(AH) \subseteq C(AG)$$

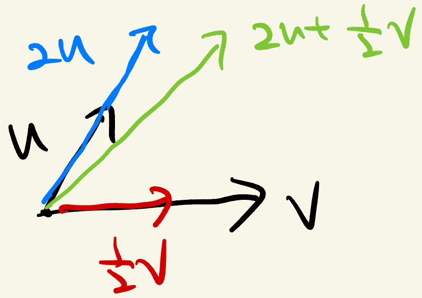
Interchanging G, H $\Rightarrow C(AG) \subseteq C(AH)$ #.

Geometric meaning of span:

(I)



III



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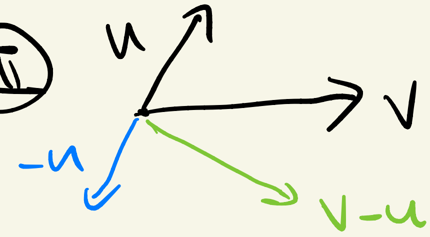

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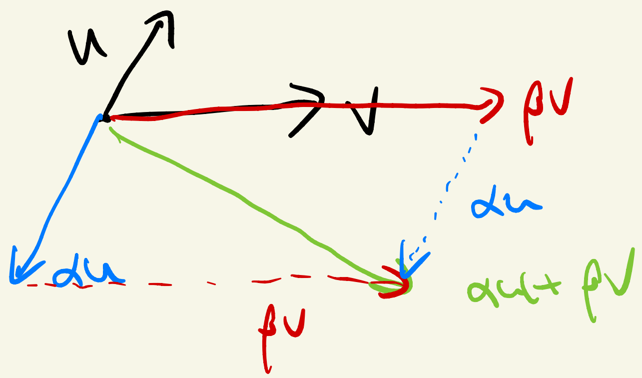
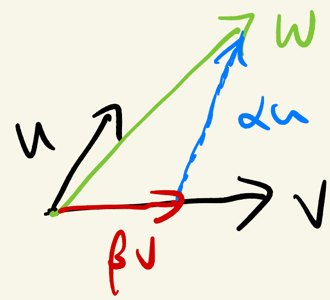


<u>Generally</u>:

$\therefore$ Any vector $w \in \mathbb{R}^2$ is inside $\mathrm{span}\{v, u\}$.



$w = \alpha u + \beta v$.

Relation to $LS(A, b)$

Given $m \times n$ matrix A, column vector $b \in \mathbb{R}^m$

Solving $Ax = b$ is equivalent to find $x_1, \dots, x_n \in \mathbb{R}$

s.t. $$[u_1 | u_2 | \cdots | u_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

i.e. $$\sum_{i=1}^{n} \underline{\overset{\in\mathbb{R}}{x_i}}\, \underline{\overset{\in\mathbb{R}^m}{u_i}} = b \in \mathbb{R}^m.$$

And so is equivalent to check if $b \in C(A)$.

Linear independence: aim to find the minimal "generators" (called basis).

$S = \{u_1, \cdots, u_n\} \in \mathbb{R}^m$ is linearly indep.

if there are no non-trivial $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ s.t.

$$\sum_{i=1}^{n} x_i u_i = 0 \quad \left(\text{And equivalently, } Null(A) = 0\right)$$

$\boxed{\text{to determine linear indep. or Not: make use of } LS(A, 0).}$

Eg: $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly indep.

consider $[A|0] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = RREF.$

$$\therefore \quad Null(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\therefore$ If $\exists \lambda_i$ s.t. $\sum_{i=1}^{4} \lambda_i v_i = 0$, then each λ_i must be 0.

(i.e. there are no non-trivial sol. to $\sum_{i=1}^{4} x_i v_i = 0$)

Similarly, if in RREF of $[A]$, there are some free column, then $Null(A) \neq \{0\}$ and hence S = linearly dependent.

Goal: Given $S \subseteq \mathbb{R}^m$, find the "minimal" subset $\hat{S} \subseteq S$

s.t. $\text{span}(S) = \text{span}(\hat{S})$.

Thm: Let $S = \{u_1, u_2, \cdots, u_n\}$, $\bar{S} = \{v_1, \cdots, v_k\} \subseteq \mathbb{R}^m$ s.t.

① For each $i = 1, 2, \cdots, n$, $u_i \in \text{span}(\bar{S})$

② For each $j = 1, 2, \cdots, k$, $v_j \in \text{span}(S)$,

then $\text{span}(\bar{S}) = \text{span}(S)$.

pf: By ①, $\exists \lambda_{ij} \in \mathbb{R}$ s.t. $u_i = \sum_{j=1}^{k} \lambda_{ij} v_j$, for each i.

Let $y \in \text{span}(S)$, $\exists \alpha_i \in \mathbb{R}$ s.t.

$$y = \sum_{i=1}^{n} \alpha_i u_i = \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{k} \lambda_{ij} v_j \right)$$

$$= \sum_{j=1}^{k} \underbrace{\left(\sum_{i=1}^{n} \alpha_i \lambda_{ij} \right)}_{\in \mathbb{R}} v_j \in \text{span}(\bar{S}).$$

$\therefore \text{span}(S) \subseteq \text{span}(\bar{S})$.

Interchanging S and $\bar{S}$ above $\Rightarrow \text{span}(\bar{S}) \subseteq \text{span}(S)$ #

Thm, Let $S = \{u_1, u_2, \cdots, u_n\} \subseteq \mathbb{R}^m$ be a linearly independent subset. Then $\bar{S} = S \cup \{v\}$ is linearly dependent iff $v \in \text{span}(S)$.

pf: ($\Leftarrow$): $v = \sum_{i=1}^{n} \alpha_i u_i$ for some $\alpha_i \in \mathbb{R}$.

where some $\alpha_i \neq 0$ otherwise $v = 0$.

$\therefore \exists$ non-trivial $\{\lambda_i\}_{i=1}^{n+1}$ s.t.

$\sum_{i=1}^{n} \lambda_i u_i + \lambda_{n+1} v = 0$. $\therefore$ linear dependent.

($\Rightarrow$): $\exists \{\lambda_i\}_{i=1}^{n+1}$ s.t. $\sum_{i=1}^{n} \lambda_i u_i + \lambda_{n+1} v = 0$

s.t. some $\lambda_i \neq 0$.

If $\lambda_{n+1} = 0$, then $\sum_{i=1}^{n} \lambda_i u_i = 0$

$\Rightarrow \lambda_i \equiv 0 \rightarrow\leftarrow$

$\therefore \lambda_{n+1} \neq 0 \Rightarrow v = -\sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1}} u_i \in \text{span}(S)$.

Defn: Let $V \subseteq \mathbb{R}^m$ be a subspace which is not zero subspace (i.e. $V \neq \{0\}$)

A subset $S = \{v_1, v_2, \cdots, v_n\}$ is said to be a basis for V if ① S is linear indep.

② $V = \text{span } S$.

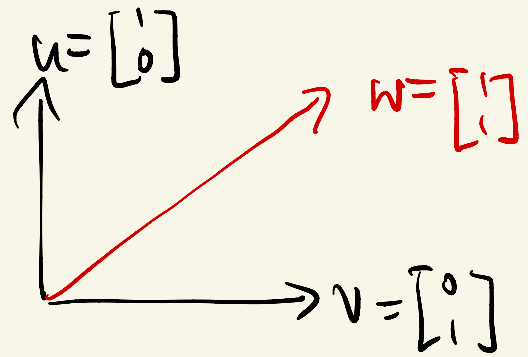
convention: if $V = \{0\}$, basis $= \phi$

Eg: $\mathbb{R}^m \subseteq \mathbb{R}^m$. then $S = \{e_1, \cdots, e_m\}$ where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th row} \quad \left(= E^{n,1}_{i,1} \right)$$

is a basis for $\mathbb{R}^m$.

Eg:



- $S = \{u, v\}$ is a basis for $\mathbb{R}^2$.
- $\bar{S} = S \cup \{w\}$ also span $\mathbb{R}^2$

Any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ can be expressed as

- $xu + yv \in \text{span}(S) = \mathbb{R}^2$.
- $(x-t)u + (y-t)v + tw$ where $t \in \mathbb{R}$.

  $\in \text{span}(\bar{S}) = \mathbb{R}^2$. freedom (too much infor.)

Thm: Let V be subspace of $\mathbb{R}^m$, $S = \{v_1, \cdots, v_n\} \subseteq V$.

then S = basis for V iff $\forall x \in V$, $\exists!\, \alpha_i \in \mathbb{R}$ s.t.

$$x = \sum_{i=1}^{n} \alpha_i v_i.$$

(Unique coeff.)

pf: $(\Rightarrow)$: If $x = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i$

then $\quad 0 = \sum_{i=1}^{n} (\alpha_i - \beta_i) v_i$

$\Rightarrow \alpha_i = \beta_i \quad \forall i = 1, 2, \cdots, n$ by linear indep. #

$(\Leftarrow)$: ① $\text{span}(S) = V$

② S is linearly indep. since

- $0 \in V$, $\exists! \ \alpha_i \in \mathbb{R}$ s.t. $\sum_{i=1}^{n} \alpha_i v_i = 0$

- $0 = \sum_{i=1}^{n} 0 \cdot v_i$

$\Rightarrow \alpha_i = 0 \quad \forall i = 1, 2, \cdots, n$ #.

Thm Suppose $V \subseteq \mathbb{R}^n$ is a subspace with S = basis for V, then Number of elements in $S \leq n$.

pf: let $S = \{u_1, \cdots, u_m\} \subseteq \mathbb{R}^n$ be a basis.

Then S is linearly indep.

If $m > n$, then $[u_1 \ u_2 \cdots u_m | 0]$ is row equivalent to $[A | 0]$ with 0 (linearly indep.) free column

$\Rightarrow A = I_m$ which is impossible.

RMK: a subspace V can have more than one set of basis!!

Thm (Existence of basis)

Given a subspace $V \subseteq \mathbb{R}^n$, then V has a basis S with no. of elements ≥ 1, $\leq n$.

pf: Take $u_1 \neq 0 \in V$, $S_1 = \{u_1\}$

- If $V = \text{span } S_1$, done.
- Otherwise, $\exists\, u_2 \in V \setminus \text{span}\{u_1\}$.

$\Rightarrow S_2 = \{u_1, u_2\}$ is linearly independent since otherwise $u_2 \in \text{span}\{u_1\} \rightarrow\leftarrow$.

- If $V = \text{span } S_2$, done.
- otherwise $\exists\, u_3 \neq 0 \in V \setminus \text{span}(S_2)$.

$\Rightarrow S_3 = \{u_1, u_2, u_3\}$ is linearly independent.

Inductively, it stops at the m-th step and (since $\leq n$) obtain $S_m = \{u_1, u_2, \cdots, u_m\}$ where

$V = \text{span}(S_m)$. #

Thm (pf to be done later)

Any two basis for a subspace $V \subseteq \mathbb{R}^2$ has same numbers of elements.

The number is called $\dim(V)$.

Goal: finding basis for null space of A.

Example:

(I). $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$. find $\mathrm{Null}(A) = \{x \in \mathbb{R}^3 \mid Ax = 0\}$.

($\Leftrightarrow$ solve $Ax=0$)

$$A \longrightarrow A' = \mathrm{RREF} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\therefore \mathrm{Null}(A) = \left\{ \begin{bmatrix} -2t \\ 3t \\ -4t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

(II) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix}$. find $\mathrm{Null}(A) = \{x \in \mathbb{R}^3 \mid Ax = 0\}$

$$A \longrightarrow A' = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{RREF}.$$

$$\therefore \mathrm{Null}(A) = \left\{ \begin{bmatrix} 3t - 2s \\ -2t + s \\ s \\ t \end{bmatrix} ; s, t \in \mathbb{R} \right\}$$

$$= \mathrm{span}\left\{ \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

general

Recall: the number of pivot column in RREF = rank(A).

Thm. Let A be $m \times n$ matrix, A' be the RREF of A.

Suppose $\mathrm{rank}(A') = r$. Label the pivot columns of A' as $d_1, d_2, \cdots, d_r$

Label the free columns of A' as $f_1, f_2, \cdots, f_{n-r}$.

For each $h = 1, 2, \cdots, r$ and $k = 1, 2, \cdots, n-r$;

denote s_{hk} = the (d_h, f_k)-th entry of A'.

e.g: $\left(= \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$

For each $k = 1, 2, \cdots, n-r$,

$\left(\begin{matrix} d_1 = 1 & f_1 = 3 \\ d_2 = 2 & f_2 = 4 \end{matrix} \right)$

define $u_k =$ vector in $\mathbb{R}^n$ whose
- f_k-th entry $= 1$.
- f_j-th entry $= 0$ if $j \neq k$.
- d_h-th entry $= -s_{hk}$.

Then $\{u_1, u_2, \cdots, u_{n-r}\}$ is a basis for $Null(A)$.

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pf: Step 1: $Au_k = 0$ s.t. $u_k \in Null(A)$

Step 2: $\{u_1, \cdots, u_{n-r}\}$ = linear indep.

Step 3: $Null(A) = span\{u_1, \cdots, u_{n-r}\}$.

Step 1: $Au_k = 0 \iff A'u_k = 0$ (since $\exists$ invertible H s.t. $HA = A'$.)

Note: $u_{k,l}$ = the l-th entry of u_k

$$= \begin{cases} -s_{hk} & \text{if } l = d_h, \ 1 \le h \le r \\ 1 & \text{if } l = f_k \\ 0 & \text{if } l = f_j, \ j \neq k. \end{cases}$$

eg: $A' = \begin{bmatrix} 1 & 0 & \boxed{2} & \boxed{-3} \\ 0 & 1 & \boxed{-1} & \boxed{2} \\ 0 & 0 & \boxed{0} & \boxed{0} \end{bmatrix}$, $u_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

$A'u_k = m \times 1$ matrix

For $l=1,2,\cdots m$, $(A' u_k)_{l1} = \sum_{i=1}^{n} A'_{li} u_{ki}$

$$= \sum_{\substack{i=d_h \\ 1\le h\le r}} A'_{li} \cdot u_{ki} + A'_{lf_k} \cdot$$

$$= \sum_{h=1}^{r} A'_{ld_h} \cdot (-S_{hk}) + A'_{lf_k}$$

Case 1: If $l = d_{h'}$ for some $h' \in \{1, 2, \cdots, r\}$

then $= -S_{h'k} + A'_{d_{h'}f_k} = -S_{h'k} + S_{h'k} = 0.$

Case 2: If $l \neq d_h$ for any $h \in \{1, 2, \cdots, r\}$.

then $A'_{ld_h} = 0$ because

the d_h-th column of A' is pivot.

$\cdot$ $A'_{f_{k'}f_k} = 0$ by defn of free column.

$$\Rightarrow \sum_{h=1}^{r} A'_{ld_h} \cdot (-S_{hk}) + A'_{lf_k} = 0 \quad \#$$

$\therefore$ $A' u_k = 0$ for all $k=1,2,\cdots, r$.

Step 2: $\{u_1, \cdots, u_{n-r}\}$ is linearly indep. since:

If $\exists \alpha_i$ s.t. $\sum_{i=1}^{n-r} \alpha_i u_i = 0$ (as a vector in $\mathbb{R}^n$)

Consider the f_j-th entry

(Eg: $u_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$)

$$= \left(\sum_{i=1}^{n-r} \alpha_i u_i\right)_{f_j 1}$$

$$= \sum_{i=1}^{n-r} \alpha_i u_{if_j} = \alpha_j = 0 \quad \forall j = 1, 2, \cdots, n-r \ \#.$$

Step 3: $\operatorname{span}\{u_1, \cdots, u_{n-r}\} = \operatorname{Null}(A)$. ($\subseteq$: done)

Let $x \in \mathbb{R}^n$ s.t. $Ax = 0 \iff A'x = 0$ ($x \in \operatorname{Null}(A')$)

$\therefore \forall l = 1, 2, \cdots m,$

$$0 = (A'x)_{l1}$$

$$= \sum_{i=1}^{n} A'_{li} \cdot x_i$$

$$= \sum_{h=1}^{r} A'_{ld_h} \cdot x_{d_h} + \sum_{k=1}^{n-r} A'_{lf_k} \cdot x_{f_k}$$

$$= \sum_{h=1}^{r} \delta_{ld_h} \cdot x_{d_h} + \sum_{k=1}^{n-r} A'_{lf_k} \cdot x_{f_k}.$$

$\therefore$ if we fix $l = d_{h'}$, then

$$0 = x_{d_{h'}} + \sum_{k=1}^{n-r} A'_{d_{h'}f_k} \cdot x_{f_k}$$

$$= x_{d_{h'}} + \sum_{k=1}^{n-r} s_{h'k} \cdot x_{f_k}.$$

$\therefore$ for any $h = 1, 2, \cdots, r$, we have

$$x_{d_h} = - \sum_{k=1}^{n-r} s_{hk} \, x_{f_k}.$$

Since $\left(\sum_{k=1}^{n-r} x_{f_k} \cdot u_k \right)_l = \sum_{k=1}^{n-r} x_{f_k} \cdot u_{kl}$

if $l = d_h$, $= -\sum_{k=1}^{n-r} x_{f_k} \cdot s_{hk} = x_{d_h} = x_l$.

if $l = f_{k'}$, $= \sum_{k=1}^{n-r} x_{f_k} \cdot u_{k f_{k'}} = x_{f_{k'}} = x_l$.

$$\boxed{\therefore \quad \sum_{k=1}^{n-r} x_{f_k} \cdot u_k = x.}$$